Problem

Find the quadrupole moment of a distribution of charge on the surface of a sphere of radius a with surface charge density $\sigma = \sigma_0 \left(\frac{3}{2}cos^2\theta - \frac{1}{2}\right)$. Find the total energy stored in this distribution.

Solution

Quadrupole moment in spherical polars

$$\begin{aligned} Q &= \int\int\int \sigma {r'}^2 (3cos^2\theta' - 1) d^3r' \\ &= \int\int\int \sigma {r'}^2 (3cos^2\theta' - 1) {r'}^2 sin\theta' d\theta' d\varphi' dr' \\ &= \frac{1}{2}\int\int\int \sigma_0 {r'}^4 (3cos^2\theta' - 1)^2 sin\theta' d\theta' d\varphi' dr' \end{aligned}$$

The integral over φ yields a factor of 2π and $r' = a \quad \Rightarrow$

$$\begin{aligned} Q &= \sigma_0 \pi a^4 \int_0^\pi (3cos^2\theta' - 1)^2 sin\theta' d\theta' \\ &= \sigma_0 \pi a^4 \int_0^\pi (9cos^4\theta' sin\theta' - 6cos^2\theta' sin\theta' + sin\theta') d\theta' \\ &= \sigma_0 \pi a^4 \left[-\frac{9cos^5\theta'}{5} + 2cos^3\theta' - cos\theta' \right]_0^\pi \\ &= \sigma_0 \pi a^4 \left[\left(\frac{9}{5} - 2 + 1) \right) - \left(-\frac{9}{5} + 2 - 1) \right) \right] \\ &= \frac{8\sigma_0 \pi a^4}{5} \end{aligned}$$

For a *surface charge* distribution the total energy stored

$$E = \int \phi dq = \frac{1}{2}\int\int\int \phi\, \sigma d^3r'$$

Where $\phi(\mathbf{r}) = \phi_0 + \phi_1 + \phi_2 + \ldots$

Both total charge q and dipole moment d are zero, hence the monopole and dipole terms vanish, and

$$E = \frac{1}{2}\int\int\int \phi_2\, \sigma d^3r'$$

Where $\phi_2 = \dfrac{Q}{16\pi\epsilon_0 {r'}^3}(3cos^2\theta - 1) = \dfrac{\sigma_0 a^4}{10\epsilon_0 {r'}^3}(3cos^2\theta - 1)$

The integral over φ yields a factor of 2π, $r' = a$ and simplifying $\quad \Rightarrow$

$$\begin{aligned} E &= \frac{\pi {\sigma_0}^2 a^3}{20\epsilon_0} \int_0^\pi (3cos^2\theta' - 1)^2 sin\theta' d\theta' \\ &= \frac{\pi {\sigma_0}^2 a^3}{20\epsilon_0} \left(\frac{8}{5} \right) \\ &= \frac{2\pi {\sigma_0}^2 a^3}{25\epsilon_0} \end{aligned}$$
