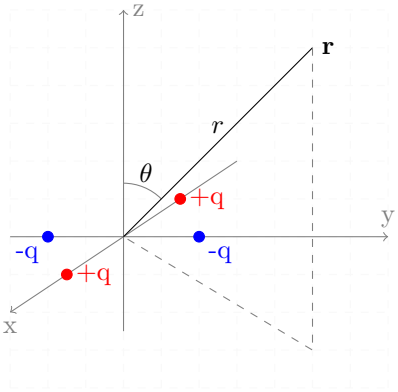
Problem

Two equal charges q are located at points $(\pm a, 0, 0)$, and two charges $-q$ at $(0, \pm a, 0)$. Find the leading term in the potential at large distances, and the corresponding electric field.

Solution

Potential for $r >> a$

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}+a\mathbf{i}|} + \frac{1}{|\mathbf{r}-a\mathbf{i}|} - \frac{1}{|\mathbf{r}+a\mathbf{j}|} - \frac{1}{|\mathbf{r}-a\mathbf{j}|}\right)$$



Where

$$|\mathbf{r}+a\mathbf{i}|^2 = r^2 + 2a\mathbf{i}\cdot\mathbf{r} + a^2 = r^2 + 2ax + a^2$$

$$\begin{aligned}
\frac{1}{|\mathbf{r}+a\mathbf{i}|} &= \frac{1}{r}\left(1 + \frac{2ax}{r^2} + \frac{a^2}{r^2}\right)^{-1/2} \\
&\approx \frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{2ax+a^2}{r^2}\right) + \frac{3}{8}\left(\frac{2ax+a^2}{r^2}\right)^2 - \ldots\right) \\
&\approx \frac{1}{r} - \frac{ax}{r^3} - \frac{a^2}{2r^3} + \frac{3}{8r}\left(\frac{4a^2x^2}{r^4}\right) \quad \text{neglecting } \mathcal{O}\,a^3 \\
&= \frac{1}{r} - \frac{ax}{r^3} + \frac{3a^2x^2 - a^2r^2}{2r^5}
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{1}{|\mathbf{r}-a\mathbf{i}|} &= \frac{1}{r} + \frac{ax}{r^3} + \frac{3a^2x^2 - a^2r^2}{2r^5} \\
\frac{1}{|\mathbf{r}+a\mathbf{j}|} &= \frac{1}{r} - \frac{ay}{r^3} + \frac{3a^2y^2 - a^2r^2}{2r^5} \\
\frac{1}{|\mathbf{r}-a\mathbf{j}|} &= \frac{1}{r} + \frac{ay}{r^3} + \frac{3a^2y^2 - a^2r^2}{2r^5}
\end{aligned}$$

Hence

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{3a^2x^2 + 3a^2x^2 - 3a^2y^2 - 3a^2y^2}{2r^5}\right) = \frac{3qa^2(x^2-y^2)}{4\pi\epsilon_0 r^5}$$

Corresponding electric field $\mathbf{E}$

$$\begin{aligned}
E_x = -\frac{\partial\phi}{\partial x} &= -\frac{3qa^2}{4\pi\epsilon_0}\frac{\partial}{\partial x}\left(\frac{x^2-y^2}{(x^2+y^2+z^2)^{5/2}}\right) \\
&= -\frac{3qa^2}{4\pi\epsilon_0}\left(\frac{(x^2+y^2+z^2)^{5/2}(2x) - (x^2-y^2)(\frac{5}{2})(x^2+y^2+z^2)^{3/2}(2x)}{(x^2+y^2+z^2)^5}\right) \\
&= -\frac{3qa^2}{4\pi\epsilon_0 r^{10}}\left(\left((x^2+y^2+z^2)^{3/2}\right)\left(2(x^2+y^2+z^2) - 5(x^2-y^2)\right)\right)x \\
&= \frac{3qa^2}{4\pi\epsilon_0 r^7}\left(3x^2 - 7y^2 - 2z^2\right)x
\end{aligned}$$

$$E_y = -\frac{\partial \phi}{\partial y} = -\frac{3qa^2}{4\pi\epsilon_0 r^7}\big(-2(x^2+y^2+z^2) - 5(x^2-y^2)\big)y$$
$$= \frac{3qa^2}{4\pi\epsilon_0 r^7}\big(7x^2 - 3y^2 + 2z^2\big)y$$

$$E_z = -\frac{\partial \phi}{\partial z} = -\frac{3qa^2}{4\pi\epsilon_0 r^7}\big(-5(x^2-y^2)\big)z$$
$$= \frac{3qa^2}{4\pi\epsilon_0 r^7}\big(5x^2 - y^2\big)z$$

$$\mathbf{E} = \frac{3qa^2}{4\pi\epsilon_0 r^7}\Big(\big(3x^2 - 7y^2 - 2z^2\big)x,\ \big(7x^2 - 3y^2 + 2z^2\big)y,\ \big(5x^2 - y^2\big)z\Big)$$
