Problem

The parallax of a star (the angle subtended at the star by the radius of the Earth's orbit) is ϖ. The star's position is observed to oscillate with angular amplitude α and period τ. If the oscillation is interpreted as being due to the existence of a planet moving in a circular orbit around the star, show that its mass m_1 is given by

$$\frac{m_1}{M} = \frac{\alpha}{\varpi}\left(\frac{M\tau_E}{M_S\tau}\right)^{2/3}$$

where M is the total mass of the star plus planet, M_S is the Sun's mass, $\tau_E = 1\,\text{year}$. Evaluate the mass m_1 if $M = 0.25M_S$, $\tau = 16\,\text{years}$, $\varpi = 0.5''$ and $\alpha = 0.01''$. What conclusion can be drawn without making the assumption that the orbit is circular?

Solution

Parallax angle of star $\varpi = a_E/r$ where $a_E = 1\,\text{AU}$ and r the star's distance from Earth.

Amplitude of star oscillation $\alpha = a_2/r$ where a_2 is the star's distance from the CM of star & planet.

Hence the star's distance from the CM of the system $a_2 = \dfrac{\alpha}{\varpi}\,a_E$

In the Star-Planet system (taking a as the planet-star distance)

$$a_2 = \frac{m_1}{M}\,a$$
$$a = \frac{M}{m_1}\frac{\alpha}{\varpi}\,a_E$$

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{1}{GM}\left(\frac{M}{m_1}\frac{\alpha}{\varpi}\right)^3 a_E^3 \qquad \text{(Kepler)}$$

In the Sun-Earth system

$$\left(\frac{\tau_E}{2\pi}\right)^2 = \frac{a_E^3}{GM_S}$$
$$= \left(\frac{\tau}{2\pi}\right)^2\left(\frac{M}{M_S}\right)\left(\frac{m_1}{M}\frac{\varpi}{\alpha}\right)^3$$
$$\left(\frac{m_1}{M}\frac{\varpi}{\alpha}\right)^3 = \left(\frac{\tau_E}{\tau}\right)^2\left(\frac{M_S}{M}\right)$$
$$\left(\frac{m_1}{M_S}\frac{\varpi}{\alpha}\right)^3 = \left(\frac{\tau_E}{\tau}\right)^2\left(\frac{M}{M_S}\right)^2$$
$$\frac{m_1}{M_S} = \frac{\alpha}{\varpi}\left(\frac{\tau_E M}{\tau M_S}\right)^{2/3}$$

$M = 0.25\,M_S \quad \tau = 16 \quad \varpi = 0.5''$ and $\alpha = 0.01''$

$$m_1 = \left(\frac{0.01}{0.5}\left(\frac{1\times 0.25}{16}\right)^{2/3}\right)M_S = 0.00125\,M_S$$

If the planet's orbit is elliptic, a would not necessarily be its major axis, hence $m_1 \geq 0.000125\,M_S$