Problem

A spherical satellite moving through a uniform tenuous atmosphere is subject to a retarding force. If the orbit of the satellite is highly elliptical, the retarding force is concentrated almost entirely in the lowest part of the orbit. Replace it by an impulsive force of impulse I delivered once every orbit, at perigee. By considering changes in energy and angular momentum, find the changes in parameters a and l. Show that $\delta l = \delta a(1-e)^2$, and hence that the effect is to decrease the period and apogee distance, while leaving the perigee distance unaffected. (The orbit therefore becomes more and more circular with time.) Show that the velocity at apogee increases, while that at perigee decreases.

Solution

$$E = -\frac{GMm}{r} + \frac{1}{2}mv^2 \qquad \text{Total energy} \qquad (1)$$

$$J = mvr \qquad \text{Angular momentum} \qquad (2)$$

$$a = \frac{GMm}{2|E|} \qquad b^2 = \frac{J^2}{2m|E|} \qquad \text{Elliptic orbit} \qquad (3)$$

$$a = \frac{l}{1-e^2} \qquad b^2 = al \qquad e^2 = 1 - \left(\frac{b}{a}\right)^2 \qquad \text{Ellipse properties} \qquad (4)$$

At perigee $r = a(1-e) \quad v = J/ma(1-e) \quad \delta v = I/m$ hence the change in energy δE is given by equation (1)

$$\begin{aligned}
\delta E &= \frac{1}{2}m\big(v^2 - (v-\delta v)^2\big) = -mv\delta v \\
&= -m\left(\frac{J}{ma(1-e)}\right)\frac{I}{m} \\
&= -\frac{JI}{ma(1-e)} \\
\delta E^2 &= \frac{J^2I^2}{m^2a^2(1-e)^2} \\
&= \frac{2I^2|E|}{m}\,\frac{b^2}{a^2(1-e)^2} \qquad \text{since} \quad J^2 = 2m|E|b^2 \\
&= \frac{2I^2|E|}{m}\,\frac{a^2(1-e^2)}{a^2(1-e)^2} \qquad \text{since} \quad b^2 = a^2(1-e^2) \\
&= \frac{2I^2|E|}{m}\,\frac{(1+e)}{(1-e)} \\
\delta E^2 &= \frac{I^2GM}{a}\,\frac{(1+e)}{(1-e)} \qquad \text{since} \quad |E| = GMm/2a
\end{aligned}$$

From (3) $\quad |E| = GMm/2a \quad \Rightarrow \quad |\delta E| = -\dfrac{GMm}{2a^2}\,\delta a$ hence subs for δE gives

$$\begin{aligned}
\left(\frac{GMm}{2a^2}\right)^2\delta a^2 &= \frac{I^2GM}{a}\,\frac{(1+e)}{(1-e)} \\
&= 4\,\frac{I^2}{m^2}\,\frac{a^3}{GM}\,\frac{(1+e)}{(1-e)} \\
\delta a &= -2\,\frac{I}{m}\sqrt{\frac{(1+e)\,a^3}{(1-e)\,GM}}
\end{aligned}$$

At perigee the change in angular momentum δJ is given by equation (2)

$$\begin{aligned}\delta J &= -m\delta v r \\ &= -Ia(1-e)\end{aligned}$$

From (3) and (4) $J^2 = 2m|E|b^2 = GMm^2 l \quad \Rightarrow \quad 2J\,\delta J = GMm^2\delta l$ hence subs for δJ gives

$$\begin{aligned}2J\big(-Ia(1-e)\big) &= GMm^2\delta l \\ \delta l &= -\frac{2JIa(1-e)}{GMm^2} \\ &= -\frac{2Ia(1-e)}{GMm^2}\sqrt{2m|E|b^2} \qquad \text{since} \quad J^2 = 2m|E|b^2 \\ &= -\frac{2Ia(1-e)}{GMm}\sqrt{\frac{GMb^2}{a}} \qquad \text{since} \quad |E| = GMm/2a \\ &= -2\frac{I}{m}(1-e)\sqrt{\frac{ab^2}{GM}} \\ \delta l &= -2\frac{I}{m}(1-e)\sqrt{\frac{a^3(1-e^2)}{GM}} \qquad \text{since} \quad b^2 = a^2(1-e^2)\end{aligned}$$

$$\begin{aligned}\delta l &= -2\frac{I}{m}(1-e)\sqrt{\frac{a^3(1-e^2)}{GM}} \\ &= -2\frac{I}{m}(1-e)^2\sqrt{\frac{a^3(1-e)(1+e)}{GM(1-e)^2}} \\ \delta l &= \delta a(1-e)^2\end{aligned}$$

Since the impulse is given at the perigee and is in the direction $-v$ the perigee remains unaffected, but the parameters $a,\ b,\ e$ and l are changed.

Since a has decreased, the period according to Kepler's Law $\tau \propto a^{3/2}$ will decrease and the apogee will also decrease since the semi-major axis $(2a)$ has decreased, whilst the perigee has remained unchanged.

Since the orbit is becoming more circular over time the parameter e is also decreasing. Within any particular orbit J is conserved, the velocities at apogee and perigee are thus related $v_a(1+e) = v_p(1-e)$. Hence as e decreases v_a must increase whilst v_p decreases i.e. the velocity at apogee increases each orbit.
