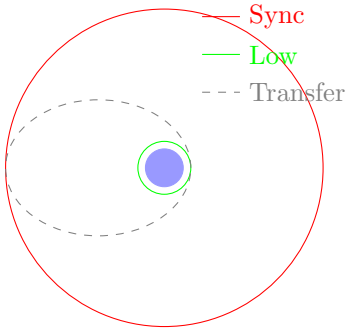
Problem

A satellite is orbiting the Earth in a circular orbit $230\,km$ above the equator. Calculate the total velocity impulse needed to place it in a synchronous orbit, using a semi-elliptical transfer orbit which just touches both circles. Given that the final mass to be placed in orbit is $30\,kg$, and the ejection velocity of the rocket is $2.5\,km\,s^{-1}$, find the necessary initial mass.

Solution

For a semi-elliptical transfer the satellite requires a velocity impulse in low orbit, which is at the perigee of it's transfer path and a further impulse at it's apogee to match the velocity required for a synchronous orbit.



It's distance r_1 from the Earth's centre in low orbit is $6608\,km$ $(6378+230)$ and it's required distance r_2 for a synchronous orbit is given by Kepler's Law

$$\frac{(r_2)^3}{GM_E} = \left(\frac{\tau}{2\pi}\right)^2$$

$$(r_2)^3 = GM_E\left(\frac{1\,day}{2\pi}\right)^2$$

$$= 3.986\times10^{14}\left(\frac{24\times60\times60}{2\pi}\right)^2$$

$$r_2 = 4.2241\times10^7\,m \quad \text{or} \quad 42241\,km$$

Hence the major axis $(2a)$ for the transfer ellipse is $48850\,km$ $(42241+6608)$ which is sufficient to calculate the required the velocity at perigee v_p and resulting velocity v_a at apogee.

For an elliptical orbit, the total energy is given by

$$E = \frac{k}{2a} = -\frac{GM_Em}{2a} \qquad \text{hence}$$

$$\frac{1}{2}mv_p^2 - \frac{GM_Em}{r_1} = -\frac{GM_Em}{2a} \qquad \text{(perigee)}$$

$$v_p^2 = 2GM_E\left(\frac{1}{r_1}-\frac{1}{2a}\right)$$

$$= 2(3.986\times10^{14})\left(\frac{1}{6.608\times10^6}-\frac{1}{4.885\times10^7}\right)$$

$$= 1.043\times10^8 \qquad \text{or} \qquad v_p = 1.021\times10^4\,m\,s^{-1}$$

$$v_p = 10.21\,km\,s^{-1}$$

$$\frac{1}{2}mv_a^2 - \frac{GM_Em}{r_2} = -\frac{GM_Em}{2a} \qquad \text{(apogee)}$$

$$= 2(3.986\times10^{14})\left(\frac{1}{4.224\times10^7}-\frac{1}{4.885\times10^7}\right)$$

$$= 2.554\times10^6 \qquad \text{or} \qquad v_a = 1.598\times10^3\,m\,s^{-1}$$

$$v_a = 1.598\,km\,s^{-1}$$

The velocities for low orbit v_1 and for synchronous orbit v_2 are given by the relation $2T = |V|$

$$
\begin{aligned}
mv_1^2 &= \frac{GM_E m}{r_1} \\
v_1^2 &= \frac{3.986 \times 10^{14}}{6.608 \times 10^6} = 6.032 \times 10^7 \qquad \text{or} \qquad v_1 = 7.767 \times 10^3 \, m\, s^{-1} \\
v_1 &= 7.767 \, km\, s^{-1}
\end{aligned}
$$

$$
\begin{aligned}
v_2^2 &= \frac{3.986 \times 10^{14}}{4.224 \times 10^7} = 9.437 \times 10^6 \qquad \text{or} \qquad v_2 = 3.071 \times 10^3 \, m\, s^{-1} \\
v_2 &= 3.071 \, km\, s^{-1}
\end{aligned}
$$

Hence the total velocity impulse is given by $(v_p - v_1) + (v_2 - v_a) = 2.44 + 1.47 = 3.91 \, km\, s^{-1}$

The mass M_2 before the second velocity impulse is given by

$$
\begin{aligned}
M &= M_2 \, e^{-(v_2 - v_a)/u} \\
30 &= M_2 \, e^{-(1.47/2.5)} \\
M_2 &= 30(1.80) = 54.0 \, kg
\end{aligned}
$$

The mass M_1 before the first velocity impulse is given by

$$
\begin{aligned}
M_2 &= M_1 \, e^{-(v_p - v_1)/u} \\
54.0 &= M_1 \, e^{-(2.44/2.5)} \\
M_1 &= 54(2.65) = 143.3 \, kg
\end{aligned}
$$

The necessary initial mass is 143 kg

Note: This can be done in a single step i.e. $30 = M_0 \, e^{-(3.91/2.5)} \quad \rightarrow \quad M_0 = 143.3 \, kg.$