Problem

If the residual mass of a rocket, without payload or fuel, is a given fraction λ of the initial mass including fuel (but still without payload). The total take-off mass required to accelerate a payload m to velocity v with an ejection velocity u is

$$M_0 = m\frac{1-\lambda}{e^{-v/u}-\lambda}.$$

Find an analogous formula for a two-stage rocket, in which each stage produces the same velocity impulse. (The first-stage rocket is discarded when its fuel is burnt out.) If $\lambda = 0.15$ and the ejection velocity is 2.5 $km\,s^{-1}$, what is the minimum number of stages required to reach escape velocity ($11.2\,km\,s^{-1}$)? With that number, what take-off mass is required, if the payload mass is $100\,kg$?

Solution

For the second stage, velocity impulse v

$$M_1 = m\,\frac{1-\lambda}{e^{-v/u}-\lambda}$$

Where M_1 is the total mass the end of first stage.

For the first stage, also velocity impulse v, the total take-off mass M_0 is given by

$$M_0 = M_1\,\frac{1-\lambda}{e^{-v/u}-\lambda}$$
$$M_0 = m\left(\frac{1-\lambda}{e^{-v/u}-\lambda}\right)^2$$

Note: v in this formula is the velocity impulse per stage (assumed equal).

The upper limit to the velocity impulse for each stage with the given parameters is $4.74\,km\,s^{-1}$ i.e. when $e^{v/u}-\lambda = 0$. Hence to attain an escape velocity of $11.2\,km\,s^{-1}$ requires a minimum of 3 stages.

If each stage gives an equal velocity impulse v of $11.2/3$ $km\,s^{-1}$, and $u = 2.5\,km\,s^{-1}$

$$\begin{aligned}
M_0 &= m\left(\frac{1-\lambda}{e^{-v/u}-\lambda}\right)^3 \\
&= 100\left(\frac{1-0.15}{e^{-11.2/7.5}-0.15}\right)^3 \\
&= 100\left(\frac{0.85}{.0746}\right)^3 \\
&= 100\times 11.39^3 \\
&= 147800\,kg
\end{aligned}$$

The required take-off mass for a payload of $100\,kg$ is $1.48\times 10^5\,kg$.
