Problem

Find the gain in kinetic energy when a rocket emits a small amount of matter. Hence calculate the total energy which must be supplied from chemical or other sources to accelerate the rocket to a given velocity. Show that this is equal to the energy required if an equal amount of matter is ejected while the rocket is held fixed on a test-bed.

Solution



Before *After*

Consider the emission of a small mass dm $(=-dM)$ as shown. Conservation of momentum has previously given the result $M\,dv = u\,dm$ leading to the equation $M = M_0\,e^{-v/u}$ where M_0 is the initial mass.

Gain in kinetic energy dT (neglecting 2^{nd} order infinitesimals) is given by

$$\begin{aligned}
dT &= \Big((M-dm)(v-dv)^2 \,+\, u\,dm(u-v)^2 \,-\, Mv^2\Big)/\,2 \\
&= (Mv^2 + 2Mv\,dv - v^2\,dm + u^2\,dm - 2uv\,dm + v^2\,dm - Mv^2)/\,2 \\
&= (2Mv\,dv + u^2\,dm - 2uv\,dm)/\,2 \\
&= \Big(2Mv\,dv + (u^2-2uv)\frac{M}{u}\,dv\Big)/\,2 \qquad \text{since } M\,dv = u\,dm \\
&= \frac{Mu}{2}\,dv \\
&= \frac{M_0 u}{2}\,e^{-v/u}\,dv \qquad \text{hence to accelerate to } v \\
T &= \frac{M_0 u}{2}\int_0^v e^{-v/u}\,dv \\
&= \frac{M_0 u}{2}\Big[-u\,e^{-v/u}\Big]_0^v \\
&= \frac{M_0 u^2}{2}\,(1-e^{-v/u})
\end{aligned}$$

The total energy which must be supplied is $\frac{1}{2}\,M_0u^2(1-e^{-v/u})$.

If the rocket is fixed on a test-bed

$$\begin{aligned}
dT &= \frac{u^2}{2}\,dm \\
&= -\frac{u^2}{2}\,dM \qquad \text{since } dm = -dM \\
T &= -\frac{u^2}{2}\int_{M_0}^{M} dM \\
&= \frac{u^2}{2}(M_0 - M) \\
&= \frac{M_0 u^2}{2}\,(1-e^{-v/u})
\end{aligned}$$

This energy is equal to that required to accelerate the rocket if the same amount of matter is ejected.