Problem

A rocket with take-off mass M_0 is launched vertically upward from the surface of the Earth. Consider the effect of a finite burn-up time. Show that, if the rocket ejects matter at a constant rate a, then its height at time t is

$$z = ut - \frac{uM}{a}\, ln\, \frac{M_0}{M} - \tfrac{1}{2}gt^2, \qquad \text{with} \qquad M = M_0 - at.$$

Hence, show that if it burns out after time t_1, leaving a final mass M_1, then [provided that $t_1 < (u/g)\, ln(M_0/M_1)$], the maximum height reached is

$$z = \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - ut_1\left(\frac{M_0}{M_0 - M_1}\, ln \frac{M_0}{M_1} - 1\right).$$

If $M_0 = 164\, kg$, $M_1 = 100\, kg$ and $u = 2\, km\, s^{-1}$, find the maximum height reached if the burn-up time is (a) $10\, s$, and (b) $30\, s$.

Solution

$$\begin{aligned} M &= M_0\, e^{-v/u} \\ a &= -\frac{dM}{dt} \quad = \frac{1}{u}\, M_0\, e^{-v/u}\, \frac{dv}{dt} \\ a &= \frac{M}{u}\, \frac{dv}{dt} \\ \frac{dv}{dt} &= \frac{au}{M} \end{aligned}$$

Hence the rocket is subject to a total acceleration of $\dfrac{au}{M} - g$ and since $M = M_0 - at$

$$\begin{aligned} \frac{d^2z}{dt^2} &= \frac{au}{M_0 - at} - g \\ \frac{dz}{dt} &= -u\, ln(M_0 - at) - gt + c_1 \end{aligned}$$

Initial conditions: $t = 0 \quad dz/dt = 0 \quad \Rightarrow \quad c_1 = u\, ln\, M_0$

$$\begin{aligned} \frac{dz}{dt} &= -u\, (ln(M_0 - at) - ln\, M_0) - gt \\ &= -u\, ln\left(1 - \frac{a}{M_0}t\right) - gt \\ z &= -u\left(\left(t - \frac{M_0}{a}\right) ln\left(1 - \frac{a}{M_0}t\right) - t\right) - \frac{gt^2}{2} + c_2 \\ &= ut - u\left(-\frac{M}{a}\, ln\, \frac{M}{M_0}\right) - \frac{gt^2}{2} + c_2 \\ &= ut - u\left(\frac{M}{a}\, ln\, \frac{M_0}{M}\right) - \frac{gt^2}{2} + c_2 \\ &= ut - \frac{uM}{a}\, ln\, \frac{M_0}{M} - \frac{gt^2}{2} + c_2 \end{aligned}$$

Initial conditions: $t = 0 \quad z = 0 \quad M = M_0 \quad \Rightarrow \quad c_2 = 0$

Height of rocket at time t is $ut - \dfrac{uM}{a}\, ln\, \dfrac{M_0}{M} - \tfrac{1}{2}gt^2$ with $M = M_0 - at$

Height reached during burn t_1 is given by

$$z_1 = ut_1 - \frac{uM_1}{a}\, ln\, \frac{M_0}{M_1} - \frac{g(t_1)^2}{2}$$

After burn rocket has an upward velocity v given by

$$\begin{aligned} v &= -u\, ln\left(1 - \frac{a}{M_0}t_1\right) - gt_1 \\ &= u\, ln\, \frac{M_0}{M_1} - gt_1 \qquad\qquad \text{since } M_1 = M_0 - at_1 \end{aligned}$$

Note: $t_1 < (u/g)\, ln(M_0/M_1) \Rightarrow v > 0$

Hence, using conservation of energy, additional height z_2 is given by

$$\begin{aligned} 2g\, z_2 &= v^2 \\ z_2 &= \left(u\, ln\, \frac{M_0}{M_1} - gt_1\right)^2 / 2g \\ &= \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - u\, ln\, \frac{M_0}{M_1}t_1 + \frac{g(t_1)^2}{2} \end{aligned}$$

Maximum height reached is simply $z_1 + z_2$

$$\begin{aligned} z &= ut_1 - \frac{uM_1}{a}\, ln\, \frac{M_0}{M_1} + \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - u\, ln\, \frac{M_0}{M_1}t_1 \\ &= \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - \frac{uM_1}{a}\, ln\, \frac{M_0}{M_1} - u\, ln\, \frac{M_0}{M_1}t_1 + ut_1 \\ &= \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - ut_1\Big(\frac{M_1}{at_1}\, ln\, \frac{M_0}{M_1} + ln\, \frac{M_0}{M_1} - 1\Big) \\ &= \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - ut_1\left(\Big(\frac{M_1}{at_1} + 1\Big)\, ln\, \frac{M_0}{M_1} - 1\right) \\ &= \frac{u^2}{2g}\left(ln\, \frac{M_0}{M_1}\right)^2 - ut_1\Big(\frac{M_0}{M_0 - M_1}\, ln\, \frac{M_0}{M_1} - 1\Big) \end{aligned}$$

(a) $t_1 = 10s$

$$\begin{aligned} z &= \frac{(2\times 10^3)^2}{2g}\left(ln\, \frac{164}{100}\right)^2 - 2\times 10^3 \times 10\Big(\frac{164}{64}\, ln\, \frac{164}{100} - 1\Big) \\ &= 49910 - 5353 = 44560\, m \end{aligned}$$

Maximum height reached is $44.6\, km$

(b) $t_1 = 30s$

$$\begin{aligned} z &= \frac{(2\times 10^3)^2}{2g}\left(ln\, \frac{164}{100}\right)^2 - 2\times 10^3 \times 30\Big(\frac{164}{64}\, ln\, \frac{164}{100} - 1\Big) \\ &= 49910 - 16060 = 33850\, m \end{aligned}$$

Maximum height reached is $33.9\, km$