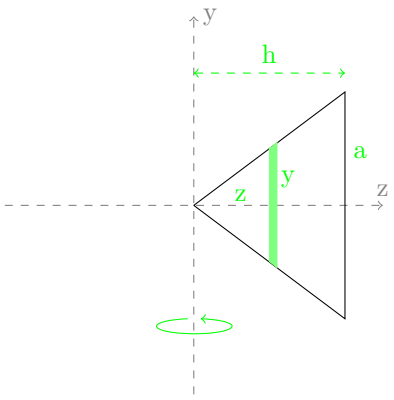
Problem

Calculate the principal moments of inertia of a uniform, solid cone of vertical height h, and base radius a about its vertex. For what value of the ratio h/a is every axis through the vertex a principal axis? For this case, find the position of the centre of mass and the principal moments of inertia about it.

Solution

The moment of inertia of a thin disc mass M and radius a about its diameter is $\frac{1}{4}Ma^2$, hence using the parallel axis theorem the moment of inertia of the disc dm about the vertex of the cone is



$$dI = \tfrac{1}{4}dm\,y^2 + dm\,z^2$$

Where $dm = \dfrac{M}{\frac{1}{3}\pi a^2 h}\,\pi y^2\,dz = \dfrac{3M}{a^2 h}y^2\,dz$

Hence

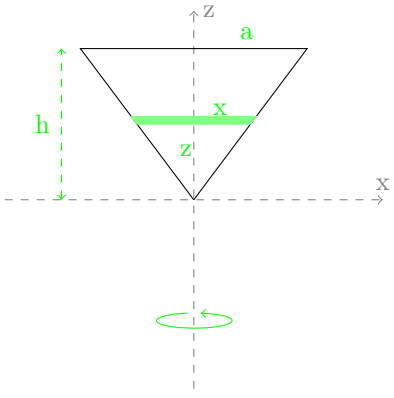
$$dI = \frac{3M}{4a^2 h}\,(y^4 + 4y^2 z^2)\,dz$$

Since $y = \dfrac{az}{h}$

$$\begin{aligned} I &= \frac{3M}{4a^2 h}\int_0^h \left(\left(\frac{a}{h}\right)^4 z^4 + 4\left(\frac{a}{h}\right)^2 z^4\right) dz \\ &= \frac{3M}{20a^2 h}(a^4 h + 4a^2 h^3) \\ &= \frac{3M}{20}(a^2 + 4h^2) \end{aligned}$$

This is the moment of inertia about both the x-axis and y-axis i.e. I_1 and I_2.

The moment of a thin disc about its central axis is $\frac{1}{2}Ma^2$, hence the moment of dm about the vertex is



$$dI = \tfrac{1}{2}\,dmx^2$$

Where $dm = \dfrac{3M}{a^2 h}x^2\,dz$

$$dI = \frac{3M}{2a^2 h}\,x^4\,dz$$

Since $x = \dfrac{az}{h}$

$$\begin{aligned} I &= \frac{3M}{2a^2 h}\left(\frac{a}{h}\right)^4 \int_0^h z^4\,dz \\ &= \frac{3Ma^2}{10} \end{aligned}$$

This is the moment of inertia about the z-axis i.e. I_3.

For the moments of inertia through the vertex are to be equal

$$\begin{aligned}\frac{3M}{20}(a^2+4h^2) &= \frac{3Ma^2}{10}\\ a^2+4h^2 &= 2a^2\\ h/a &= \tfrac{1}{2}\end{aligned}$$

The centre of mass (on the z-axis) is given by

$$\begin{aligned}z_{CM} &= \frac{1}{M}\int_0^h z\,dm\\ &= \frac{1}{M}\int_0^h z\frac{3M}{a^2h}x^2\,dz\\ &= 3\int_0^h \frac{z^3}{h^3}\,dz\\ &= \frac{3h}{4}\end{aligned}$$

Moments of inertia about the centre of mass (parallel axis theorem)

$$\begin{aligned}I_1^* = I_2^* &= I - M\left(\frac{3h}{4}\right)^2\\ &= \frac{3M}{20}(a^2+4h^2) - M\left(\frac{3h}{4}\right)^2\\ &= \frac{3M}{20}(2a^2) - M\left(\frac{3a}{8}\right)^2\\ &= \frac{6Ma^2}{20} - \frac{9Ma^2}{64}\\ &= \frac{51Ma^2}{320}\end{aligned}$$

$$I_3^* = I_3 = \frac{3Ma^2}{10}$$
