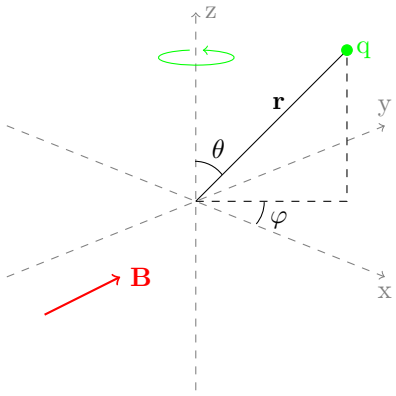
Problem

A uniformly charged sphere is spinning freely with angular velocity $\boldsymbol{\omega}$ in a uniform magnetic field $\mathbf{B}$. Taking the z-axis in the direction of $\boldsymbol{\omega}$, and $\mathbf{B}$ in the xz-plane, write down the moment about the centre of the magnetic force on a particle at $\mathbf{r}$. Evaluate the total moment of the magnetic force on the sphere, and show that it is equal to $(q/2M)\,\mathbf{J}\times\mathbf{B}$, where q and M are the total charge and mass respectively. Hence show that the axis will precess around the direction of the magnetic field with a precessional angular velocity equal to the Larmor frequency ($\omega_L = qB/2M$). What difference would it make if the charge distribution were spherically symmetric but non-uniform?

Solution

For a particle charge q, velocity $\mathbf{v}$ at $\mathbf{r}$, taking B in the direction of the y-axis, the force on the particle is given by

$$\mathbf{F} = q\,\mathbf{v}\times\mathbf{B} = -qvBsin\varphi\,\mathbf{k}$$



The moment about the centre of the sphere

$$\begin{aligned}\dot{\mathbf{J}} &= \mathbf{r}\times\mathbf{F}\\ &= qvBrsin\theta sin\varphi\,\hat{\boldsymbol{\varphi}}\\ &= qB\omega r^2 sin^2\theta sin\varphi\,\hat{\boldsymbol{\varphi}}\end{aligned}$$

Since $v = \omega r sin\theta$.

And where $\hat{\boldsymbol{\varphi}} = -sin\varphi\,\mathbf{i} + cos\varphi\,\mathbf{j}$

A sphere of radius a and uniform charge density ρ will therefore experience a total moment of magnetic force given by

$$\begin{aligned}\dot{\mathbf{J}} &= \rho B\omega\int_V r^2 sin^2\theta sin\varphi\,\hat{\boldsymbol{\varphi}}\,dV\\ &= \rho B\omega\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 sin^2\theta sin\varphi\,\hat{\boldsymbol{\varphi}}\,(r^2 sin\theta\,dr d\theta d\varphi)\\ &= \rho B\omega\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4 sin^3\theta sin\varphi\,\hat{\boldsymbol{\varphi}}\,dr d\theta d\varphi\\ &= \frac{\rho B\omega a^5}{5}\int_0^{2\pi}\int_0^{\pi} sin^3\theta sin\varphi\,\hat{\boldsymbol{\varphi}}\,d\theta d\varphi\\ &= \frac{\rho B\omega a^5}{5}\int_0^{2\pi}\left[\frac{cos\,3\theta - 9cos\,\theta}{12}\right]_0^{\pi} sin\varphi\,\hat{\boldsymbol{\varphi}}\,d\varphi\\ &= \frac{\rho B\omega a^5}{5}\int_0^{2\pi}\frac{4}{3}\,sin\varphi\,\hat{\boldsymbol{\varphi}}\,d\varphi\\ &= \frac{4\rho B\omega a^5}{15}\int_0^{2\pi} sin\varphi(-sin\varphi\,\mathbf{i} + cos\varphi\,\mathbf{j})\,d\varphi\\ &= \frac{2\rho B\omega a^5}{15}\int_0^{2\pi}(cos2\varphi - 1)\,\mathbf{i} + sin2\varphi\,\mathbf{j})\,d\varphi\end{aligned}$$

$$= \frac{2\rho B\omega a^5}{15}\left[\left(\frac{sin\,2\varphi}{2}-\varphi\right)\mathbf{i}-\frac{cos\,2\varphi}{2}\mathbf{j}\right]_0^{2\pi}$$
$$= -\frac{4\pi qB\omega a^5}{15}\,\mathbf{i}$$

Since $\rho = \frac{q}{\frac{4}{3}\pi a^3}$

$$\dot{\mathbf{J}} = -\frac{qB\omega a^2}{5}\,\mathbf{i}$$

Since $J = I\omega = \frac{2}{5}Ma^2\omega$

$$\dot{\mathbf{J}} = -\frac{qBJ}{2M}\,\mathbf{i}$$
$$= \frac{q}{2M}\,\mathbf{J}\times\mathbf{B}$$

Hence the sphere will continue to rotate with angular velocity ω, but will precess around the magnetic field with angular velocity $\omega_L = \frac{qB}{2M}$.

If the charge distribution remains spherically symmetric but is non-uniform the sphere will still precess, but with a different angular velocity than the Lamor frequency.