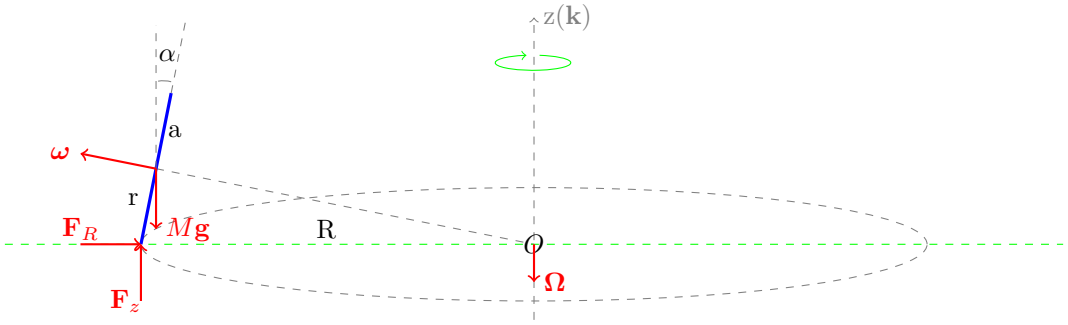
Problem

A wheel of radius a, with its mass concentrated on the rim, is rolling with velocity v round a circle of radius $R(>> a)$, maintaining a constant inclination α to the vertical. Show that $v = a\omega = R\Omega$, where ω is the angular velocity of the wheel about its axis, and $\Omega(<< \omega)$ is the precessional angular velocity of the axis. Use the momentum equation to find the horizontal and vertical components of the force at the point of contact. Then show from the angular momentum equation about the centre of mass that $R = 2v^2/g\,tan\alpha$. Evaluate R for $v = 5\,m\,s^{-1}$ and $\alpha = 30°$.

Solution



Time taken for one revolution of the wheel

$$\tau_0 = \frac{2\pi}{\omega} = \frac{2\pi a}{v} \quad \Rightarrow \quad v = a\omega$$

During this time the wheel travels a distance of $2\pi a$ around the circumference of the circle, hence time taken for a complete circle

$$\tau_1 = \frac{2\pi}{\Omega} = \tau_0 \times \frac{2\pi R}{2\pi a} = \frac{2\pi}{\omega}\frac{R}{a} \quad \Rightarrow \quad a\omega = R\Omega$$

Hence $v = a\omega = R\Omega$

The motion is characterised by a spin about its principal axis of angular velocity $\boldsymbol{\omega} = \omega\mathbf{e}_3$ and by the precession of this axis about O with angular velocity $\boldsymbol{\Omega} = -\Omega\mathbf{k}$. Since α is constant the vertical component of force at the point of contact $\mathbf{F}_z = M\mathbf{g}$. These forces act as a couple creating a change in the angular momentum of the wheel $\dot{\mathbf{J}} = \mathbf{r} \times \mathbf{F} = Mgsin\alpha\,\hat{\boldsymbol{\varphi}}$, thus turning the wheel to the right forcing it to follow a circular path. A small radial frictional force $\mathbf{F}_R \approx MR\Omega^2$ prevents the wheel from sliding out. The tangential force (not shown) is assumed negligible and sufficient only to prevent the wheel slipping round.

Angular momentum equation about the centre of mass (neglecting $\mathbf{F}_R$ since $\Omega << \omega$)

$$\dot{\mathbf{J}} = Mgsin\alpha\,\hat{\boldsymbol{\varphi}}$$

Where $\mathbf{J} = I\omega\,\mathbf{e}_3$ (moment of inertia about point of contact) $\qquad \hat{\boldsymbol{\varphi}} = (\mathbf{e}_3 \times \mathbf{k})/cos\alpha$

$$I\omega\,\dot{\mathbf{e}}_3 = Mga\,sin\alpha\,\frac{\mathbf{e}_3 \times \mathbf{k}}{cos\alpha}$$

$$\dot{\mathbf{e}}_3 = \frac{Mga\,tan\alpha}{I\omega}\,\mathbf{e}_3 \times \mathbf{k}$$

This equation describes a vector $\mathbf{e}_3$ rotating with constant angular velocity $\mathbf{\Omega}$ about the vertical, where

$$\mathbf{\Omega} = -\frac{Mga\,tan\alpha}{I\omega}\,\mathbf{k}$$

The moment of inertia (I_3) of a wheel about its axis, with its mass concentrated on the rim, is Ma^2. Hence the moment about the point of contact on the rim is $I_3 + Ma^2$ or $2Ma^2$, hence

$$\begin{aligned}
\Omega &= \frac{Mga\,tan\alpha}{2Ma^2\omega} \\
\Omega\,\omega &= \frac{g\,tan\alpha}{2a} \\
\frac{v}{R} \times \frac{v}{a} &= \frac{g\,tan\alpha}{2a} \\
R &= \frac{2v^2}{g\,tan\alpha}
\end{aligned}$$

For $v = 5\,m\,s^{-1}$ and $\alpha = 30^\circ$

$$\begin{aligned}
R &= \frac{2(5)^2}{9.81 \times tan\,30^\circ} \\
&= 8.83\,m
\end{aligned}$$
