Problem

An insect of mass $100\,mg$ is resting on the edge of a flat uniform disc of mass $3\,g$ and radius $50\,mm$, which is rotating at $60\,r.p.m.$ about a smooth pivot. The insect crawls in towards the centre of the disc. Find the angular velocity when it reaches it, and the gain in kinetic energy. Where does this kinetic energy come from, and what happens to it when the insect crawls back out to the edge?

Solution

The moment of inertia of a flat disc mass M and radius a rotating about its centre is $\frac{1}{2}Ma^2$. Angular momentum is conserved, hence

$$\tfrac{1}{2}Ma^2\omega_1 = \tfrac{1}{2}Ma^2\omega_0 + ma^2\omega_0$$

Where m is the mass of the insect and ω_1 and ω_0 are the angular velocities when the insect is at the centre and edge respectively.

$$\begin{aligned}
\omega_1 &= \omega_0\left(1+\frac{2m}{M}\right)\\
&= 60\left(1+\frac{2(1\times10^{-4})}{3\times10^{-3}}\right)\\
&= 60\left(1+\frac{2}{30}\right)\\
\omega_1 &= 64\ r.p.m.
\end{aligned}$$

Total kinetic energy when the insect is on the edge ($\omega_0 = 60\,r.p.m. = 2\pi\,rads\,s^{-1}$)

$$\begin{aligned}
T_0 &= \tfrac{1}{2}(\tfrac{1}{2}Ma^2)\omega_0^2 + \tfrac{1}{2}ma^2\omega_0^2\\
&= \frac{a^2\omega_0^2}{4}(M+2m)\\
&= \frac{(5\times10^{-2})^2(2\pi)^2}{4}\left(3\times10^{-3}+2(1\times10^{-4})\right)\\
&= (2.467\times10^{-2})(3.2\times10^{-3})\\
&= 7.896\times10^{-5}\,J
\end{aligned}$$

Total kinetic energy when the insect is at the centre ($\omega_1 = 64\,r.p.m. = 32\pi/15\,rads\,s^{-1}$)

$$\begin{aligned}
T_1 &= \tfrac{1}{2}(\tfrac{1}{2}Ma^2)\omega_1^2\\
&= \frac{Ma^2\omega_1^2}{4}\\
&= \frac{(3\times10^{-3})(5\times10^{-2})^2}{4}\left(\frac{32\pi}{15}\right)^2\\
&= (1.875\times10^{-6})(44.92)\\
&= 8.422\times10^{-5}\,J
\end{aligned}$$

Gain in kinetic energy ($T_1 - T_0$) is $5.26\times10^{-6}J$.

This energy comes from work done by the insect and is dissipated as heat on the return.