Problem

Find the principal moments of inertia of a flat rectangular plate of mass $30\,g$ and dimensions $80\,mm \times 60\,mm$. Given that the plate is rotating about a diagonal with angular velocity 15 rads s^{-1}, find the components of angular momentum parallel to the edges. Given that the axis is of total length $120\,mm$, and is held vertical by bearings at its ends, find the horizontal component of the force on each bearing.

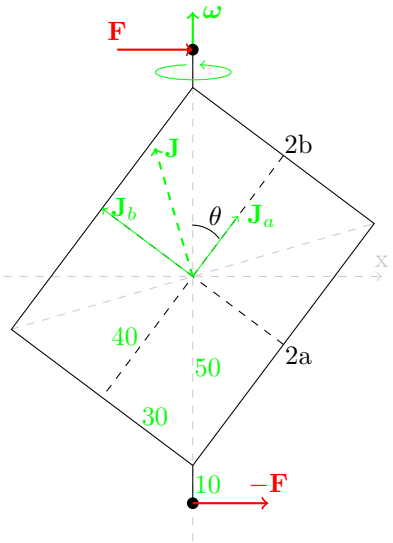
Solution

Moments of Inertia

$$I_a^* = \tfrac{1}{3}Mb^2 = \tfrac{1}{3}(.03)(.03)^2 = 9 \times 10^{-6}\,kg\,m^2$$

$$I_b^* = \tfrac{1}{3}Ma^2 = \tfrac{1}{3}(.03)(.04)^2 = 16 \times 10^{-6}\,kg\,m^2$$

$$I_c^* = \tfrac{1}{3}M(a^2+b^2) = \tfrac{1}{3}(.03)(.04^2+.03^2) = 25 \times 10^{-6}\,kg\,m^2$$



Angular momenta

$$J_a = \omega I_a^* cos\theta = 15(9 \times 10^{-6})\tfrac{4}{5} = 1.08 \times 10^{-4}\,kg\,m^2\,s^{-1}$$

$$J_b = \omega I_b^* sin\theta = 15(16 \times 10^{-6})\tfrac{3}{5} = 1.44 \times 10^{-4}\,kg\,m^2\,s^{-1}$$

$$J_c = 0$$

The resultant angular momentum **J** is at right angles to the other diagonal of the rectangle.

Horizontal component of force

The products of inertia give rise to centrifugal forces which are balanced by the couple $\mathbf{F}d$ where d is the distance between between the bearings. There are no external forces and $\boldsymbol{\omega}$ is constant, hence in the xz-plane

$$Fd = I_{xz}\omega^2$$

There are two contributions (in opposite directions) to I_{xz}, hence

$$\begin{aligned} Fd &= -\iiint \rho\, x_b z_b\, \omega^2\, d^3r + \iiint \rho\, x_a z_a\, \omega^2\, d^3r \\ &= -\iiint \rho\, r_b^2\, sin\theta cos\theta\, \omega^2\, d^3r + \iiint \rho\, r_a^2\, sin\theta cos\theta\, \omega^2\, d^3r \\ &= (-I_a^* + I_b^*)\, sin\theta cos\theta\, \omega^2 \\ 0.12\,F &= (7 \times 10^{-6})(0.6)(0.8)(15)^2 \\ F &= (7.56 \times 10^{-4})/0.12 \\ &= 6.3 \times 10^{-3}\,N \end{aligned}$$
