Problem

Find the principal moments of inertia of a uniform solid cube of mass M and edge length $2a$ (a) with respect to the mid-point of an edge, and (b) with respect to a vertex.

Solution

The moment of inertia of a cube of mass M and edge $2a$ is $\frac{2}{3}Ma^3$ about *any* principal axis. *

a) Taking z-axis through the mid-point of opposite edges and $R(a\sqrt{2})$ as the distance from a mid-point to the centre of mass, the moments of inertia about the x and y axes are given by the parallel axes theorem

$$I_{xx} = I_{yy} = \tfrac{2}{3}Ma^2 + MR^2 = \tfrac{8}{3}Ma^2$$

The z-axis is a principal axis, hence

$$I_{zz} = \tfrac{2}{3}Ma^2$$

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b) Taking z-axis through opposite vertices and $R(a\sqrt{3})$ as the distance from a vertex to the centre

$$I_{xx} = I_{yy} = \tfrac{2}{3}Ma^2 + MR^2 = \tfrac{11}{3}Ma^2$$

Again the z-axis is a principal axis, hence

$$I_{zz} = \tfrac{2}{3}Ma^2$$

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* This can be verified either by direct integration or by using Routh's rule. e.g. taking z-axis through the centre of opposite faces

$$I_{zz} = \iiint_V \rho(x^2+y^2)d^3r = \frac{M}{8a^3}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}(x^2+y^2)dxdydz = \tfrac{2}{3}Ma^2$$

or Routh's rule

$$K_{z0} = \iiint_{V_0}\zeta^2 d\xi d\eta d\zeta = 4\int_{-1}^{1}\zeta^2 d\zeta = \tfrac{8}{3}$$

Since $M_0 = 8$

$$\lambda_x = \lambda_y = \lambda_z = \tfrac{1}{3}$$

And

$$I_{zz} = M(\lambda_x a^2 + \lambda_y a^2) = \tfrac{2}{3}Ma^2$$

Similarly Routh's rule may be used if the z-axis is taken through the mid-point of opposite edges, in this case

$$K_{z0} = 4\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\zeta^2(\sqrt{2}-\zeta)d\zeta = \tfrac{8}{3}$$

i.e. the same result is true for the moment of inertia about this axis.

The same result also applies to the moment of inertia through opposite vertices i.e. the moment of inertia is the same about all 3 sets of principal axes.