Exercise 5

For a three-dimensional elastic medium, the potential energy is

$$V = \frac{\mathcal{T}}{2}\int \mathrm{d}^3x(\nabla\psi)^2$$

Where $\mathcal{T}$ represents the tension/stress; and the kinetic energy is

$$T = \frac{\rho}{2}\int \mathrm{d}^3x\left(\frac{\partial\psi}{\partial t}\right)^2$$

Where ρ represents the mass density.

Use these results, and the functional derivative approach, to show that ψ obeys the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Where v is the velocity of the wave.

Solution

The Lagrangian density is given by

$$\mathcal{L} = T - V = \frac{\rho}{2}\int \mathrm{d}^3x\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{\mathcal{T}}{2}\int \mathrm{d}^3x(\nabla\psi)^2$$

And the action principle (using the Einstein summation convention) is

$$\frac{\delta S}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0$$

Hence

$$\frac{\delta S}{\delta\psi} = 0 - \frac{\rho}{2}\left(-2\frac{\partial^2\psi}{\partial t^2}\right) + \frac{\mathcal{T}}{2}\big(-2\nabla^2\psi\big) \quad *$$

$$\nabla^2\psi = \frac{1}{\mathcal{T}/\rho}\frac{\partial^2\psi}{\partial t^2}$$

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

The wave equation with $v = \sqrt{\mathcal{T}/\rho}$.

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* See example 1.4 in the text and also result 1.19, together with the generalization: If $I = \int\left(\nabla\psi\right)^2\mathrm{d}^3x$ then $\dfrac{\delta I}{\delta\psi} = -2\nabla^2\psi$.