Exercise 6

Show that if

$$Z_0[J] = exp\left(-\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\,\Delta(x-y)J(y)\right)$$

where $\Delta(x) = \Delta(-x)$ then

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = -\left[\int \mathrm{d}^4y\,\Delta(z_1-y)J(y)\right] Z_0[J]$$

Solution

$$\begin{aligned}
\frac{\delta Z_0[J]}{\delta J(z_1)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[exp\left(-\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,[J(x)+\epsilon\delta(x-z_1)]\,\Delta(x-y)\,[J(y)+\epsilon\delta(y-z_1)]\right)\\
&\qquad -exp\left(-\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y[J(x)\Delta(x-y)]J(y)\right)\Bigg]\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[exp\Big(-\tfrac{1}{2}\Big[\int \mathrm{d}^4x\,\mathrm{d}^4yJ(x)\Delta(x-y)J(y)\\
&\qquad +\epsilon\Big(\int \mathrm{d}^4xJ(x)\Delta(x-z_1)+\int \mathrm{d}^4yJ(y)\Delta(z_1-y)\Big)+\mathcal{O}(\epsilon^2)\Big]\Big)\\
&\qquad -exp\left(-\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)\,J(y)\right)\Bigg]\\
&\approx \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[exp\Big(-\tfrac{1}{2}(\mathcal{A}+2\epsilon\mathcal{B})\Big)-exp\Big(-\tfrac{1}{2}\mathcal{A}\Big)\right] \qquad *\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[exp\big(-\tfrac{1}{2}\mathcal{A}\big)\Big(exp(-\epsilon\mathcal{B})-1\Big)\right]\\
&\approx \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[exp\big(-\tfrac{1}{2}\mathcal{A}\big)\big(-\epsilon\mathcal{B}\big)\right] \qquad \text{(expanding } exp(-\epsilon\mathcal{B})\approx 1-\epsilon\mathcal{B})\\
\frac{\delta Z_0[J]}{\delta J(z_1)} &= -Z_0[J]\left[\int \mathrm{d}^4y\,\Delta(z_1-y)J(y)\right]
\end{aligned}$$

$*\quad \mathcal{A} = \int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\,\Delta(x-y)J(y) \qquad \mathcal{B} = \int \mathrm{d}^4xJ(x)\Delta(z_1-x) = \int \mathrm{d}^4yJ(y)\Delta(z_1-y)$