Exercise 1

For a one-dimensional harmonic oscillator with annihilation and creation operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

show that

$$[\hat{a}, \hat{a}] = 0 \qquad [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \qquad [\hat{a}, \hat{a}^\dagger] = 1 \qquad \text{and} \qquad \hat{H} = \hbar\omega\big(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\big)$$

Solution

$$\begin{aligned}[\hat{a}, \hat{a}] &= \hat{a}\hat{a} - \hat{a}\hat{a} \\ &= 0 \qquad\qquad *\end{aligned}$$

$$\begin{aligned}[\hat{a}^\dagger, \hat{a}^\dagger] &= \frac{m\omega}{2\hbar}\left(-\frac{i}{m\omega}[\hat{x}, \hat{p}] - \frac{i}{m\omega}[\hat{p}, \hat{x}]\right) \quad **\\ &= 0 \qquad\qquad \text{since} \quad [\hat{x}, \hat{p}] = i\hbar = -[\hat{p}, \hat{x}]\end{aligned}$$

$$\begin{aligned}[\hat{a}, \hat{a}^\dagger] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\ &= \frac{m\omega}{2\hbar}\left(-\frac{i}{m\omega}[\hat{x}, \hat{p}] + \frac{i}{m\omega}[\hat{p}, \hat{x}]\right) \\ &= \frac{m\omega}{2\hbar}\left(\frac{\hbar}{m\omega} + \frac{\hbar}{m\omega}\right) \\ &= 1\end{aligned}$$

$$\begin{aligned}\hat{H} &= \tfrac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m} \\ &= \tfrac{1}{2}m\omega^2\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) - \frac{i\omega}{2}[\hat{x}, \hat{p}] \quad ***\\ &= \tfrac{1}{2}m\omega^2\left(\frac{2\hbar}{m\omega}\right)\hat{a}^\dagger\hat{a} + \frac{\hbar\omega}{2} \\ \hat{H} &= \hbar\omega\big(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\big)\end{aligned}$$

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* Operators of this type commute with themselves
** Writing out in full to check above statement, see example 2.1 in the text
*** See equation 2.8 in the text.