Exercise 2

For the Hamiltonian $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}^4$, where λ is small, show by writing the Hamiltonian in terms of creation and annihilation operators and using perturbation theory, that the energy eigenvalues of all the levels are given by

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega + \frac{3\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2 (2n^2 + 2n + 1)$$

Solution

For a harmonic oscillator without perturbation, the text gives

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger) \qquad\qquad \hat{p} = -i\sqrt{\frac{2m\omega}{\hbar}}\,(\hat{a} - \hat{a}^\dagger)$$

With the annihilation and creation operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \qquad\qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

The Hamiltonian has a small perturbation $\Delta\hat{H} = \lambda\hat{x}^4$, hence

$$\begin{aligned}
\Delta\hat{H} &= \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \langle n|\left(\hat{a} + \hat{a}^\dagger\right)^4 |n\rangle \\
&= \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \langle n|\left(\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger\right)|n\rangle \quad * \\
&= \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \Big((n+1)(n+2) + (n+1)^2 + (n+1)n + n(n-1) + n^2 + n(n+1)\Big) \\
&= \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \left(6n^2 + 6n + 3\right) \\
&= \frac{3\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2 (2n^2 + 2n + 1)
\end{aligned}$$

Hence the energy eigenvalues of all levels are increased by this amount and

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega + \frac{3\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2 (2n^2 + 2n + 1)$$

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* Only including the 'middle' term in the expansion since the 'active ingredients' of the Hamiltonian are the ones with combinations of $\hat{a}\hat{a}^\dagger$. The relation $\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$ together with $n = \langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle$ and $n + 1 = \langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle$ are then used.