Exercise 4

Using $\hat{a}\,|0\rangle = 0$ and the annihilation and creation operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

together with

$$\langle x|\,\hat{p}\,|\psi\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\langle x|\psi\rangle$$

show that

$$\left(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\langle x|0\rangle = 0$$

and hence

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

Solution

$$\begin{aligned}
\hat{a}\,|0\rangle &= 0 \\
\langle x|\,\hat{a}\,|0\rangle &= 0 \\
\langle x|\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|0\rangle &= 0 \\
\langle x|\,\hat{x}\,|0\rangle + \frac{i}{m\omega}\langle x|\,\hat{p}\,|0\rangle &= 0 \\
x\,\langle x|0\rangle + \frac{i}{m\omega}\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\langle x|0\rangle\right) &= 0 \\
\left(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\langle x|0\rangle &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\,\langle x|0\rangle}{\mathrm{d}x} &= -\frac{m\omega}{\hbar}x\,\langle x|0\rangle \\
\frac{\mathrm{d}\,\langle x|0\rangle}{\mathrm{d}\,\langle x|0\rangle} &= -\frac{m\omega}{\hbar}x\,dx \\
\langle x|0\rangle &= Ae^{-m\omega x^2/2\hbar} \quad \text{where } A \text{ is a constant}
\end{aligned}$$

Normalizing $\langle x|0\rangle$

$$\begin{aligned}
\int_{-\infty}^{\infty} |\,\langle x|0\rangle\,|^2 \mathrm{d}x &= 1 \\
A^2\int_{-\infty}^{\infty} e^{-x^2(m\omega/\hbar)} &= 1 \\
A^2\sqrt{\frac{\pi\hbar}{m\omega}} &= 1 \quad \rightarrow \quad A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}
\end{aligned}$$

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$