Exercise 2

Show that for a simple harmonic oscillator

(a) $\left[\hat{a}, (\hat{a}^\dagger)^n\right] = n(\hat{a}^\dagger)^{n-1}$

(b) $\langle 0|\, \hat{a}^n (\hat{a}^\dagger)^m \,|0\rangle = n!\, \delta_{nm}$

(c) $\langle m|\, \hat{a}^\dagger \,|n\rangle = \sqrt{n+1}\, \delta_{m,n+1}$

(d) $\langle m|\hat{a}|n\rangle = \sqrt{n}\, \delta_{m,n-1}$

Solution

a)

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\right] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\
&= 1(\hat{a}^\dagger)^0
\end{aligned}$$

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\hat{a}^\dagger\right] &= \hat{a}\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a} \\
&= \hat{a}\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a} \\
&= \left[\hat{a}, \hat{a}^\dagger\right]\hat{a}^\dagger + \hat{a}^\dagger\left[\hat{a}, \hat{a}^\dagger\right] \\
&= 2(\hat{a}^\dagger)^1
\end{aligned}$$

$$\begin{aligned}
\left[\hat{a}, (\hat{a}^\dagger)^n\right] &= \left[\hat{a}, \hat{a}^\dagger(\hat{a}^\dagger)^{n-1}\right] \\
&= \left[\hat{a}, \hat{a}^\dagger\right](\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger\left[\hat{a}, (\hat{a}^\dagger)^{n-1}\right] \\
&= (\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger\left[\hat{a}, \hat{a}^\dagger(\hat{a}^\dagger)^{n-2}\right] \\
&= (\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger\left(\left[\hat{a}, \hat{a}^\dagger\right](\hat{a}^\dagger)^{n-2} + (\hat{a}^\dagger)^2\left[\hat{a}, (\hat{a}^\dagger)^{n-2}\right]\right) \\
&= (\hat{a}^\dagger)^{n-1} + (\hat{a}^\dagger)^{n-1} + (\hat{a}^\dagger)^2\left[\hat{a}, (\hat{a}^\dagger)^{n-2}\right] \\
&\ldots \\
&= n(\hat{a}^\dagger)^{n-1}
\end{aligned}$$

b)

$$\begin{aligned}
\langle 0|\, \hat{a}^n (\hat{a}^\dagger)^m \,|0\rangle &= \langle n|\, \sqrt{n!}\, \sqrt{m!}\, |m\rangle \\
&= \sqrt{n!m!}\, \langle n|m\rangle \\
&= n!\, \delta_{nm}
\end{aligned}$$

c)

$$\begin{aligned}
\langle m|\, \hat{a}^\dagger \,|n\rangle &= \langle m|\, \sqrt{n+1}\, |n+1\rangle \\
&= \sqrt{n+1}\, \langle m|n+1\rangle \\
&= \sqrt{n+1}\, \delta_{m,n+1}
\end{aligned}$$

d)

$$\begin{aligned}
\langle m|\hat{a}|n\rangle &= \langle m|\, \sqrt{n}\, |n-1\rangle \\
&= \sqrt{n}\, \langle m|n-1\rangle \\
&= \sqrt{n}\, \delta_{m,n-1}
\end{aligned}$$