Exercise 3

The three-dimensional harmonic oscillator is described by the Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) + \tfrac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)$$

Define creation operators $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, and $\hat{a}_3^\dagger$ so that $\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}$ and show that $\hat{H} = \hbar\omega\sum_{i=1}^{3}(\frac{1}{2} + \hat{a}_i^\dagger\hat{a}_i)$.

Angular momentum can be defined as

$$\hat{L}^i = -i\hbar\epsilon^{ijk}\hat{a}_j^\dagger\hat{a}_k \qquad \text{where } \epsilon^{ijk} \text{ is the Levi-Civita symbol}$$

so that for example $\hat{L}^3 = -i\hbar(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1)$.

Define new creation operators

$$\hat{b}_1^\dagger = -\frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + i\hat{a}_2^\dagger) \qquad \hat{b}_0^\dagger = \hat{a}_3^\dagger \qquad \hat{b}_{-1}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger - i\hat{a}_2^\dagger)$$

and show that $\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}$, the Hamiltonian is $\hat{H} = \hbar\omega\sum_{m=-1}^{1}(\frac{1}{2} + \hat{b}_m^\dagger\hat{b}_m)$ and $\hat{L}^3 = \hbar\sum_{m=-1}^{1} m\hat{b}_m^\dagger\hat{b}_m$

Solution

Defining creation and annihilation operators so that $\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}$

$$\hat{a}_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right) \qquad \hat{a}_i = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)$$

The Hamiltonian may be written (see equations 2.11 to 2.14 in the text)

$$\hat{H} = \sum_{i=1}^{3}\frac{\hat{p}_i^2}{2m} + \tfrac{1}{2}m\omega^2\hat{x}_i^2 = \hbar\omega\sum_{i=1}^{3}\left(\tfrac{1}{2} + \hat{a}_i^\dagger\hat{a}_i\right)$$

Evaluating the commutators for the new operators

$$\begin{aligned}\left[\hat{b}_1, \hat{b}_1^\dagger\right] &= \tfrac{1}{2}\left[(\hat{a}_1 - i\hat{a}_2), (\hat{a}_1^\dagger + i\hat{a}_2^\dagger)\right] \\ &= \tfrac{1}{2}\left[(\hat{a}_1 - i\hat{a}_2)(\hat{a}_1^\dagger + i\hat{a}_2^\dagger) - (\hat{a}_1^\dagger + i\hat{a}_2^\dagger)(\hat{a}_1 - i\hat{a}_2)\right] \\ &= \tfrac{1}{2}\Big(\left[\hat{a}_1, \hat{a}_1^\dagger\right] + \left[\hat{a}_2, \hat{a}_2^\dagger\right]\Big) = 1\end{aligned}$$

Similarly $\left[\hat{b}_0, \hat{b}_0^\dagger\right] = \left[\hat{b}_{-1}, \hat{b}_{-1}^\dagger\right] = 1$

$$\begin{aligned}\left[\hat{b}_1, \hat{b}_0^\dagger\right] &= -\tfrac{1}{\sqrt{2}}\left[(\hat{a}_1 - i\hat{a}_2), \hat{a}_3^\dagger\right] = 0 \\ \left[\hat{b}_1, \hat{b}_{-1}^\dagger\right] &= \tfrac{1}{2}\Big(\left[\hat{a}_1, \hat{a}_1^\dagger\right] - \left[\hat{a}_2, \hat{a}_2^\dagger\right]\Big) = 0\end{aligned}$$

Similarly $\left[\hat{b}_i, \hat{b}_j^\dagger\right] = 0$ for all $i \neq j$

Hence $\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}$

Evaluating $\hat{b}_m^\dagger \hat{b}_m$

$$\begin{aligned}
\hat{b}_{-1}^\dagger \hat{b}_{-1} &= \tfrac{1}{2}\left((\hat{a}_1^\dagger - i\hat{a}_2^\dagger)(\hat{a}_1 + i\hat{a}_2)\right) = \tfrac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 + i\hat{a}_1^\dagger\hat{a}_2 - i\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\right) \\
\hat{b}_0^\dagger \hat{b}_0 &= \hat{a}_3^\dagger\hat{a}_3 \\
\hat{b}_1^\dagger \hat{b}_1 &= \tfrac{1}{2}\left((\hat{a}_1^\dagger + i\hat{a}_2^\dagger)(\hat{a}_1 - i\hat{a}_2)\right) = \tfrac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - i\hat{a}_1^\dagger\hat{a}_2 + i\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\right) \\
\sum_{m=-1}^{1} \hat{b}_m^\dagger \hat{b}_m &= \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + \hat{a}_3^\dagger\hat{a}_3 = \sum_{i=1}^{3} \hat{a}_i^\dagger\hat{a}_i
\end{aligned}$$

And the Hamiltonian may also be written

$$\hat{H} = \hbar\omega \sum_{m=-1}^{1} \left(\tfrac{1}{2} + \hat{b}_m^\dagger \hat{b}_m\right)$$

Angular momentum $\hat{L}^3$

$$\begin{aligned}
\hat{L}^3 &= -i\hbar(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1) \\
&= \hbar\left(-i\hat{a}_1^\dagger\hat{a}_2 + i\hat{a}_2^\dagger\hat{a}_1\right) \\
&= \hbar\left(-\hat{b}_{-1}^\dagger\hat{b}_{-1} + \hat{b}_1^\dagger\hat{b}_1\right) \\
&= \hbar \sum_{m=-1}^{1} m\hat{b}_m^\dagger\hat{b}_m
\end{aligned}$$
