**Exercise 4**

Since for fermions we have that $\hat{c}_1^\dagger \hat{c}_2^\dagger |0\rangle = -\hat{c}_2^\dagger \hat{c}_1^\dagger |0\rangle$, then the state $|11\rangle$ depends on which fermion is added first. If we put the first fermion into the state $\psi_i(r_1)$ and the second fermion into the state $\psi_j(r_2)$, a representation for $|11\rangle$ is

$$\Psi(r_1, r_2) = \frac{1}{\sqrt{2}}\Big[\psi_1(r_1)\psi_2(r_2) - \psi_2(r_1)\psi_1(r_2)\Big]$$

Show that a generalization of this result for $N$ fermions is $\Psi(r_1 \ldots r_N)$, which can be written out as

$$\frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_1(r_1) & \psi_2(r_1) & \ldots & \psi_N(r_1) \\ \psi_1(r_2) & \psi_2(r_2) & \ldots & \psi_N(r_2) \\ . & . & . & . \\ . & . & . & . \\ \psi_1(r_N) & \psi_2(r_N) & \ldots & \psi_N(r_N) \end{vmatrix}$$

This expression is known as a **Slater determinant**.

**Solution**

For the state $|111\rangle$, there are three choices for the first fermion, then two for the second, hence 3! combinations to consider for the representation, hence

$$\begin{aligned}\Psi(r_1, r_2, r_3) = \frac{1}{\sqrt{3!}}\Big[&\psi_1(r_1)\psi_2(r_2)\psi_3(r_3) - \psi_1(r_1)\psi_2(r_3)\psi_3(r_2) \\ &\psi_1(r_2)\psi_2(r_1)\psi_3(r_3) - \psi_1(r_2)\psi_2(r_3)\psi_3(r_1) \\ &\psi_1(r_3)\psi_2(r_1)\psi_3(r_2) - \psi_1(r_3)\psi_2(r_2)\psi_3(r_1)\Big]\end{aligned}$$

The factor $1/\sqrt{3!}$ is for normalization.

This may be written

$$\Psi(r_1, r_2, r_3) = \frac{1}{\sqrt{3!}}\begin{vmatrix} \psi_1(r_1) & \psi_2(r_1) & \psi_3(r_1) \\ \psi_1(r_2) & \psi_2(r_2) & \psi_3(r_2) \\ \psi_1(r_3) & \psi_2(r_3) & \psi_3(r_3) \end{vmatrix}$$

Hence in the general case

$$\Psi(r_1, r_2 \ldots r_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_1(r_1) & \psi_2(r_1) & \ldots & \psi_N(r_1) \\ \psi_1(r_2) & \psi_2(r_2) & \ldots & \psi_N(r_2) \\ . & . & . & . \\ . & . & . & . \\ \psi_1(r_N) & \psi_2(r_N) & \ldots & \psi_N(r_N) \end{vmatrix}$$