**Exercise 1**

We can define a generalized commutator $\left[\hat{A},\hat{B}\right]_\zeta$ as

$$\left[\hat{A},\hat{B}\right]_\zeta = \hat{A}\hat{B} - \zeta\hat{B}\hat{A}$$

so that $\zeta = 1$ yields $\left[\hat{A},\hat{B}\right]_\zeta = \left[\hat{A},\hat{B}\right]$ for bosons and $\zeta = -1$ yields $\left[\hat{A},\hat{B}\right]_\zeta = \left\{\hat{A},\hat{B}\right\}$ for fermions, the generalized commutation relations can then be written

$$\left[\hat{\psi}(\boldsymbol{x}),\hat{\psi}^\dagger(\boldsymbol{y})\right]_\zeta = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$$

and

$$\left[\hat{\psi}(\boldsymbol{x}),\hat{\psi}(\boldsymbol{y})\right]_\zeta = 0$$

Using the argument given in the text (§4.3*) and the generalized commutator show that $\hat{V}_{wrong}$ yields the same result for both bosons and fermions.

**Solution**

For a two particle interaction, the potential energy operator for normal ordering

$$\hat{V} = \tfrac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{y})V(\mathbf{x},\mathbf{y})\hat{\psi}(\boldsymbol{y})\hat{\psi}(\boldsymbol{x})$$

Noting from the generalized commutation relations

$$\hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{y}) = \zeta\hat{\psi}^\dagger(\boldsymbol{y})\hat{\psi}(\boldsymbol{x}) + \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \qquad \text{and} \qquad \hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{y}) = \zeta\hat{\psi}(\boldsymbol{y})\hat{\psi}(\boldsymbol{x})$$

Consider the following expression (not normal ordered)

$$\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{y})\hat{\psi}(\boldsymbol{y})$$

Then swapping adjacent pairs of operators (twice) for normal ordering

$$\begin{aligned}
&\zeta\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{y})\hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{y}) + \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{y}) \quad \text{(swap 2 \& 3)}\\
&\zeta^2\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{y})\hat{\psi}(\boldsymbol{y})\hat{\psi}(\boldsymbol{x}) + \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{y}) \quad \text{(swap 3 \& 4)}
\end{aligned}$$

Since $\zeta^2$ is the the same for both fermions and bosons i.e 1; not using normal ordering yields the same incorrect result

$$\hat{V}_{wrong} = \hat{V} + \tfrac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,V(\boldsymbol{x},\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})$$

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* This section concerns **normal ordering** of particles in a two particle interaction. Creation operators are to the left and annihilation operators to the right. The book also adopts the additional convention for ordering of coordinates right-to-left $x, y, y, x$. Normal ordering ensures that the operator $\hat{V}$ has zero vacuum expectation value.