Exercise 2

Show that Poisson brackets anticommute

$$\{A, B\}_{PB} = -\{B, A\}_{PB}$$

and also satisfy the Jacobi identity

$$\{\{A, B\}_{PB}, C\}_{PB} + \{\{C, A\}_{PB}, B\}_{PB} + \{\{B, C\}_{PB}, A\}_{PB} = 0$$

and show that the quantum mechanical commutators also have the same properties.

Solution

$$\begin{aligned}\{A, B\}_{PB} &= \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \\ &= -\left(\frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i}\right) \\ &= -\{B, A\}_{PB}\end{aligned}$$

$$\begin{aligned}\{\{A, B\}_{PB}, C\}_{PB} &= \frac{\partial}{\partial q_i}\{A, B\}_{PB}\frac{\partial C}{\partial p_i} - \frac{\partial}{\partial p_i}\{A, B\}_{PB}\frac{\partial C}{\partial q_i} \\ &= \frac{\partial}{\partial q_i}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)\frac{\partial C}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)\frac{\partial C}{\partial q_i} \\ &= \left(\frac{\partial^2 A}{\partial q_i^2}\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial p_i} + \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial q_i \partial p_i}\frac{\partial C}{\partial p_i}\right) - \left(\frac{\partial^2 A}{\partial q_i \partial p_i}\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} + \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial q_i^2}\frac{\partial C}{\partial p_i}\right) \\ &- \left(\frac{\partial^2 A}{\partial q_i \partial p_i}\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_i} + \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial p_i^2}\frac{\partial C}{\partial q_i}\right) + \left(\frac{\partial^2 A}{\partial p_i^2}\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial q_i} + \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial q_i \partial p_i}\frac{\partial C}{\partial q_i}\right)\end{aligned}$$

$$\begin{aligned}\{\{C, A\}_{PB}, B\}_{PB} &= \left(\frac{\partial^2 C}{\partial q_i^2}\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial p_i} + \frac{\partial C}{\partial q_i}\frac{\partial^2 A}{\partial q_i \partial p_i}\frac{\partial B}{\partial p_i}\right) - \left(\frac{\partial^2 C}{\partial q_i \partial p_i}\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} + \frac{\partial C}{\partial p_i}\frac{\partial^2 A}{\partial q_i^2}\frac{\partial B}{\partial p_i}\right) \\ &- \left(\frac{\partial^2 C}{\partial q_i \partial p_i}\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} + \frac{\partial C}{\partial q_i}\frac{\partial^2 A}{\partial p_i^2}\frac{\partial B}{\partial q_i}\right) + \left(\frac{\partial^2 C}{\partial p_i^2}\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial q_i} + \frac{\partial C}{\partial p_i}\frac{\partial^2 A}{\partial q_i \partial p_i}\frac{\partial B}{\partial q_i}\right)\end{aligned}$$

$$\begin{aligned}\{\{B, C\}_{PB}, A\}_{PB} &= \left(\frac{\partial^2 B}{\partial q_i^2}\frac{\partial C}{\partial p_i}\frac{\partial A}{\partial p_i} + \frac{\partial B}{\partial q_i}\frac{\partial^2 C}{\partial q_i \partial p_i}\frac{\partial A}{\partial p_i}\right) - \left(\frac{\partial^2 B}{\partial q_i \partial p_i}\frac{\partial C}{\partial q_i}\frac{\partial A}{\partial p_i} + \frac{\partial B}{\partial p_i}\frac{\partial^2 C}{\partial q_i^2}\frac{\partial A}{\partial p_i}\right) \\ &- \left(\frac{\partial^2 B}{\partial q_i \partial p_i}\frac{\partial C}{\partial p_i}\frac{\partial A}{\partial q_i} + \frac{\partial B}{\partial q_i}\frac{\partial^2 C}{\partial p_i^2}\frac{\partial A}{\partial q_i}\right) + \left(\frac{\partial^2 B}{\partial p_i^2}\frac{\partial C}{\partial q_i}\frac{\partial A}{\partial q_i} + \frac{\partial B}{\partial p_i}\frac{\partial^2 C}{\partial q_i \partial p_i}\frac{\partial A}{\partial q_i}\right)\end{aligned}$$

Hence

$$\{\{A, B\}_{PB}, C\}_{PB} + \{\{C, A\}_{PB}, B\}_{PB} + \{\{B, C\}_{PB}, A\}_{PB} = 0$$

The above classical results readily crossover to the equivalent quantum commutator properties since

$$\{A, B\}_{PB} \to \frac{1}{i\hbar}\langle[\hat{A}, \hat{B}]\rangle$$

(See § 5.1 in the text.)
