Exercise 6

Use the Lagrangian

$$L = -\frac{mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} \ - qV$$

for a free particle of charge q and mass m in an electromagnetic field to derive the Lorentz force, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$\Big[$Hint: $\boldsymbol{\nabla}(\mathbf{a}\cdot\ \mathbf{b}) = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + \mathbf{b}\times \mathit{curl}\,\mathbf{a} + \mathbf{a}\times \mathit{curl}\,\mathbf{b}\Big]$

Solution

Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{x}} = 0$$

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{v}} &= \frac{\partial}{\partial \mathbf{v}}\left(-mc^2(1-v^2/c^2)^{1/2} + q\mathbf{A}\cdot\mathbf{v}\right)\\ &= \gamma m\mathbf{v} + q\mathbf{A}\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{x}} &= q\Big((\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{v}) - \boldsymbol{\nabla}V\Big)\\ &= q\Big(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) - \boldsymbol{\nabla}V\Big) \qquad *\\ &= q\Big(\mathbf{v}\times\mathbf{B} - \boldsymbol{\nabla}V\Big)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\gamma m\mathbf{v} + q\mathbf{A}\Big) &= q\Big(\mathbf{v}\times\mathbf{B} - \boldsymbol{\nabla}V\Big)\\ \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) &= q\left(-\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}V + \mathbf{v}\times\mathbf{B}\right)\\ &= q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\end{aligned}$$

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* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$