Exercise 9

Show that $\partial_\mu F^{\mu\nu} = J^\nu$ reduces to two of Maxwell's equations, and that the other two can be generated from the equation

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

Solution

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

For Maxwell's equations in free space $\partial_\mu F^{\mu\nu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) F^{\mu\nu} = (\rho, \mathbf{J})$

$$\begin{aligned} \partial_0 F^{00} + \partial_i F^{i0} &= \rho \\ \partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 &= \rho \qquad (1) \end{aligned}$$

$$\begin{aligned} \partial_0 F^{i0} + \partial_i F^{ij} &= J^j \\ -\dot{E}^1 + \partial_2 B^3 - \partial_3 B^2 &= J^1 \qquad (2a) \\ -\dot{E}^2 - \partial_1 B^3 + \partial_3 B^1 &= J^2 \qquad (2b) \\ -\dot{E}^3 + \partial_1 B^2 - \partial_2 B^1 &= J^3 \qquad (2c) \end{aligned}$$

Equation (1) $\rightarrow$ $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$ and equations (2) $\rightarrow$ $\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \dot{\mathbf{E}}$

$$F_{\mu\nu} = g_{\mu\nu} F^{\nu\mu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

For Maxwell's other two equations $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

$$\begin{aligned} \lambda = k \rightarrow \quad \partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} &= 0 \\ \partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} &= 0 \\ \partial_3 B^3 + \partial_1 B^1 + \partial_2 B^2 &= 0 \qquad (3) \end{aligned}$$

$$\begin{aligned} \lambda = 0 \rightarrow \quad \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} &= 0 \\ i = 1, j = 2 \rightarrow \quad \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} &= 0 \\ -\dot{B}^3 - \partial_1 E^2 + \partial_2 E^1 &= 0 \qquad (4a) \\ i = 2, j = 3 \rightarrow \quad \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} &= 0 \\ -\dot{B}^1 - \partial_2 E^3 + \partial_3 E^2 &= 0 \qquad (4b) \\ i = 3, j = 1 \rightarrow \quad \partial_0 F_{31} + \partial_3 F_{10} + \partial_1 F_{03} &= 0 \\ -\dot{B}^2 - \partial_3 E^1 + \partial_1 E^3 &= 0 \qquad (4c) \end{aligned}$$

Equation (3) $\rightarrow$ $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and equations (4) $\rightarrow$ $\dot{\mathbf{B}} + \boldsymbol{\nabla} \times \mathbf{E} = 0$