Exercise 1

Show that the Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{2}\big(\partial_\mu \phi\big)^2 - \frac{1}{2}m^2\phi^2$$

yields the Klein-Gordon equation when using the Euler-Lagrange equation. Also derive expressions for the momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ and the Hamiltonian density $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ for this Lagrangian.

Solution

$$\partial\mathcal{L}/\partial\phi = -m^2\phi \qquad\qquad \partial\mathcal{L}/\partial(\partial_\mu\phi) = \partial^\mu\phi \quad \text{since} \quad \big(\partial_\mu\phi\big)^2 = \partial^\mu\phi\,\partial_\mu\phi$$

Euler-Lagrange

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) &= 0 \\ -m^2\phi - \partial_\mu(\partial^\mu\phi) &= 0 \\ (\partial^2 + m^2)\phi &= 0 \qquad\qquad \text{(Klein-Gordon)} \end{aligned}$$

Momentum

$$\begin{aligned} \pi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \\ &= \frac{\partial}{\partial\dot{\phi}}\left(\frac{1}{2}\big(\partial_\mu\phi\big)^2 - \frac{1}{2}m^2\phi^2\right) \\ &= \frac{\partial}{\partial\dot{\phi}}\left(\frac{1}{2}\big(\dot{\phi}^2 - (\boldsymbol{\nabla}\phi)^2\big) - \frac{1}{2}m^2\phi^2\right) \\ &= \dot{\phi} \end{aligned}$$

Hamiltonian

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} - \mathcal{L} \\ &= \dot{\phi}^2 - \left(\frac{1}{2}\big(\partial_\mu\phi\big)^2 - \frac{1}{2}m^2\phi^2\right) \\ &= \dot{\phi}^2 - \left(\frac{1}{2}\big(\dot{\phi}^2 - (\boldsymbol{\nabla}\phi)^2\big) - \frac{1}{2}m^2\phi^2\right) \\ &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \end{aligned}$$
