Exercise 4

Show that the equation

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2$$

is equivalent to

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 - \frac{1}{2}m^2\phi^2$$

Show that

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}$$

and

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2$$

Define the quantity

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

and show that $\Pi^\mu = \partial^\mu \phi$ and $\Pi^0 = \pi = \dot{\phi}$

Solution

$$\begin{aligned}
\left(\partial_\mu \phi\right)^2 &= \dot{\phi}^2 - (\boldsymbol{\nabla}\phi)^2 \\
\mathcal{L} &= \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 \\
&= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 - \frac{1}{2}m^2\phi^2
\end{aligned}$$

$$\begin{aligned}
\pi &= \frac{\partial L}{\partial \dot{\phi}} \\
&= \dot{\phi}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H} &= \dot{\phi}^2 - \mathcal{L} \\
&= \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 - \frac{1}{2}m^2\phi^2\right) \\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2
\end{aligned}$$

$$\begin{aligned}
\Pi^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \\
&= \partial^\mu \phi
\end{aligned}$$

$$\begin{aligned}
\Pi^0 &= \partial^0 \phi = \dot{\phi} \\
&= \pi
\end{aligned}$$
